Resit Metric& topological spaces

Lenie (H.M.) Goossens S4349113

03-02-2022

Problem 1:

Decide if the function

$$d(x,y) = \begin{cases} 0 \text{ if } x = y\\ x + y \text{ if } x \neq y \end{cases}$$

is or is not a metric on the set $\mathbb{N}_{\geq 1}$, $\mathbb{N}_{\geq 0}$, $\mathbb{N}_{\geq 0} \cup \{0\}$, on $\mathbb{R}_{\geq 0}$. If 'yes' at least once, then draw the respective open disks $B_{r=10}(x_0=2)$ and $B_{r=2}(x_0=10)$

Solution:

Note that d(x, y) is a metric on the set M if:

1) $\forall x, y \in M : d(x, y) \ge 0$ and $d(x, y) = 0 \Leftrightarrow x = y$ 2) $\forall x, y \in M : d(x, y) = d(y, x)$ 3) $\forall x, y, z \in M : d(x, z) \le d(x, y) + d(y, z)$

Deff Open ball: $B_r(x_0) = \{x \in M : d(x, x_0) < r\}$

The set N>1

Let x = y. We see that d(x, y) = 0.

Take $x \neq y$. So then we see that d(x,y) = x + y. Since $x \geq 1, y \geq 1$ we can conclude that $x + y \ge 2$ hence we have that $x \ne y \Rightarrow d(x, y) > 0$.

So for all values $x, y \in \mathbb{N}_{\geq 1}$ it follows that $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$ hence the first condition holds.

Let x = y so then we have that d(x, y) = 0 and d(y, x) = 0.

Take $x \neq y$. So then we have d(x, y) = x + y and d(y, x) = y + x. Since x + y = y + x it follows that d(x, y) = d(y, x).

Hence it follows, that $\forall x,y \in \mathbb{N}_{\geq 1}$ we have d(x,y)=d(y,x) so the second condition holds.

Let x=z. So then we have that d(x,z)=0. Since $d(x,y)\geq 0$ and $d(y,z)\geq 0$ (see first condition), it follows that $d(x,y)+d(y,z)\geq 0$. So, we have that $d(x,z)\leq 0$ d(x,y) + d(y,z). Note that this holds when x = z = y and $x = z \neq y$. Let $x \neq z \neq y$. So then we have

> $d(x,z) \le d(x,y) + d(y,z)$ $\Leftrightarrow x + z \le (x + y) + (y + z) = x + 2y + z$ $\Leftrightarrow 0 \le 2y$ or Zero.

Since we have $y \in \mathbb{N}_{\geq 1}$ we know that y > 0 so 2y > 0. Therefore it is lows that d(x, y) + d(y, y) when x = (x, y)d(x, y) + d(y, z) when $x \neq y \neq z$.

Hence it follows, that $\forall x, y, z \in \mathbb{N}_{\geq 1}$ we have $d(x, z) \leq d(x, y) + d(y, z)$ so the third

condition holds. Not yet. The problem of the state of th

Since I use latex, I can not draw the open balls. But I am able to describe these open balls, by using the definition above:

 $B_{r=10}(x_0=2) = \{x \in \mathbb{N}_{\geq 1} : d(x,2) < 10\} = \{1,2,3,4,5,6,7\}$ $B_{r=2}(x_0 = 10) = \{x \in \mathbb{N}_{\geq 1} : d(x, 10) < 2\} = \{10\}$

This last open ball follows from the fact that $d(x_0, x_0) = 0 < r$

9 x + z = y

The set $\mathbb{N}_{\geq 0}$

Let x = y. We see that d(x, y) = 0.

Take $x \neq y$. So then we see that d(x,y) = x + y. Since $x \geq 0, y \geq 0$ but they are not simulatenous zero, we can conclude that $x + y \geq 1$ hence we have that $x \neq y \Rightarrow d(x,y) > 0$.

So for all values $x, y \in \mathbb{N}_{\geq 0}$ it follows that $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$ hence the first condition holds.

Let x = y so then we have that d(x, y) = 0 and d(y, x) = 0.

Take $x \neq y$. So then we have d(x, y) = x + y and d(y, x) = y + x. Since x + y = y + x it follows that d(x, y) = d(y, x).

Hence it follows, that $\forall x, y \in \mathbb{N}_{\geq 0}$ we have d(x, y) = d(y, x) so the second condition holds.

Let x=z. So then we have that d(x,z)=0. Since $d(x,y)\geq 0$ and $d(y,z)\geq 0$ (see first condition), it follows that $d(x,y)+d(y,z)\geq 0$. So,we have that $d(x,z)\leq d(x,y)+d(y,z)$. Note that this holds when x=z=y and $x=z\neq y$. Let $x\neq z\neq y$. So then we have

$$d(x,z) \leq d(x,y) + d(y,z)$$

$$\Leftrightarrow x+z \leq (x+y) + (y+z) = x+2y+z$$

$$\Leftrightarrow 0 \leq 2y$$
 not always. Since we have $y \in \mathbb{N}_{\geq 0}$ we know that $y \geq 0$ so $2y \geq 0$. Therefore it follows that $d(x,z) \leq 2y$

Since we have $y \in \mathbb{N}_{\geq 0}$ we know that $y \geq 0$ so $2y \geq 0$. Therefore it follows that $d(x,z) \leq d(x,y) + d(y,z)$ when $x \neq y \neq z$. Hence it follows, that $\forall x,y,z \in \mathbb{N}_{\geq 0}$ we have $d(x,z) \leq d(x,y) + d(y,z)$ so the third condition holds.

Since all the three condition holds, we can conclude that the function d(x, y) is a metric on the set $\mathbb{N}_{\geq 0}$.

Since I use latex, I can not draw the open balls. But I am able to describe these open balls, by using the definition above:

$$B_{r=10}(x_0 = 2) = \{x \in \mathbb{N}_{\geq 0} : d(x,2) < 10\} = \{0,1,2,3,4,5,6,7\}$$

 $B_{r=2}(x_0 = 10) = \{x \in \mathbb{N}_{\geq 0} : d(x,10) < 2\} = \{10\}$
This last open ball follows from the fact that $d(x_0, x_0) = 0 < r$

? x≠ = y.

The set $\mathbb{R}_{>0}$

Let x = y. We see that d(x, y) = 0.

Take $x \neq y$. So then we see that d(x,y) = x + y. Since $x \geq 0, y \geq 0$ but they are not simulatenous zero, we can conclude that $x+y\geq 1$ hence we have that $x\neq y\Rightarrow$ d(x,y) > 0.

So for all values $x, y \in \mathbb{R}_{\geq 0}$ it follows that $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$ hence the first condition holds.

Let x = y so then we have that d(x, y) = 0 and d(y, x) = 0.

Take $x \neq y$. So then we have d(x, y) = x + y and d(y, x) = y + x. Since x + y = y + x it follows that d(x, y) = d(y, x).

Hence it follows, that $\forall x, y \in \mathbb{R}_{\geq 0}$ we have d(x,y) = d(y,x) so the second condition holds.

Let x=z. So then we have that d(x,z)=0. Since $d(x,y)\geq 0$ and $d(y,z)\geq 0$ (see first condition), it follows that $d(x,y) + d(y,z) \geq 0$. So, we have that $d(x,z) \leq 0$. d(x,y) + d(y,z). Note that this holds when x = z = y and $x = z \neq y$. x=2=0 x=2=0 x=2=0 Let $x \neq z \neq y$. So then we have

$$d(x,z) \le d(x,y) + d(y,z)$$

$$\Leftrightarrow x + z \le (x+y) + (y+z) = x + 2y + z$$

$$\Leftrightarrow 0 \le 2y$$

 $\Leftrightarrow x+z \leq (x+y)+(y+z)=x+2y+z$ $\Leftrightarrow 0 \leq 2y$ $\Leftrightarrow 0 \leq 2y$ Since we have $y \in \mathbb{R}_{\geq 0}$ we know that $y \geq 0$ so $2y \geq 0$. Therefore it follows that $d(x,z) \leq y$ d(x, y) + d(y, z) when $x \neq y \neq z$.

Here it follows, that $\forall x, y, z \in \mathbb{R}_{\geq 0}$ we have $d(x, z) \leq d(x, y) + d(y, z)$ so the third condition hote not yet.

Since all the three condition holds we conclude that the function d(x,y) is a metric on the set $\mathbb{R}_{>0}$.

Since I use latex, I can not draw the open balls. But I am able to describe these open balls, by using the definition above:

$$B_{r=10}(x_0=2) = \{x \in \mathbb{R}_{\geq 0} : d(x,2) < 10\} = \{0,8\}$$

$$B_{r=2}(x_0=10) = \{x \in \mathbb{R}_{\geq 0} : d(x,10) < 2\} = \{10\}$$
This last open ball follows from the fact that $d(x_0,x_0) = 0 < r$

9 x = = y

1, = 3×2 pt + 1.3 + 2×4 pt = 17 pt

Resit Metric & topological spaces, University of Groningen H.M. (Lenie) Goossens

The set $\mathbb{N}_{\geq 0} \cup \{-1\}$

Take x = -1 and y = 1 (otherway around is also possible). Since $x \neq y$ it follows from the function that d(x,y) = x + y = -1 + 1 = 0. So we have that not $x, y \in \mathbb{N}_{\geq 0} \cup \{-1\}$ it holds that $d(x,y) = 0 \Leftrightarrow x = y$.

Therefore, the first condition does not hold. Since all the three conditions must hold, we can conclude that our function d(x,y) is not a metric on the set $\mathbb{N}_{\geq 0} \cup \{-1\}$

Problem 2:

Question a:

Show by a counterexample that the image f(V) of a closed set $V \subseteq X$ under a continuous map $f: X \to Y$ is not necessarily closed in Y.

Solution:

Take $X = \mathbb{R}$ with $V = [0, 2\pi]$. Let $f(x) = \tan(x - \pi/2) = \frac{\sin(x - \pi/2)}{\cos(x - \pi/2)} = \frac{\pi}{\cos(x - \pi/2)}$ We see that this function has a limit, so is not closed, so Y is not closed.

Question b:

Show by a counterexample that the image f(U) of an open set $U \subseteq X$ under a continuous map $f: X \to Y$ is not necessarily open in Y.

Solution:

Let X be the set of real numbers Take U to be an open subset of X. Let $x \in U$. Define $f: X \to Y$ such that

Y such that $f(x) = \begin{cases} 0 \text{ if } x < 0 \\ 1 \text{ if } x \ge 0 \end{cases}$

Our set of U is open while our set $f(U) \in \{0,1\}$ which is obviously not open. We see that f(x) is continuous, since for every $y \in f(U)$ there exists at least one $x \in U$ such that f(x) = y.

the writer topology?

Tso what ?

date: 03-02-2022 (S4349113)

Page 5

x=77 € [0, 2]

The set I not necessaryly countable

Problem 4:

Let (X, d_x) be a metic space and $\{U_i | i \in I\}$ be a family of connected subsets $U_i \subseteq X$ such that $U_i \cap U_j \neq \emptyset$ for all $i, j \in I$. Prove that the union $U = \bigcup_{i \in I} U_i$ is connected.

Solution:

Since U_i is not connected, we know that there does not exists open sets V_i , $W_i \subseteq U_i$ such that $U_i = V_i \cup W_i$.

I will prove that this also holds for U, by induction

Base case: $I = \{1, 2\}$.

We know that $\{U_i|i\in I\}$ is a family of connected subsets. Therefore, we have that

$$\mathbb{P}_{1}, W_{1} \subseteq U_{1} : U_{1} = V_{1} \cup W_{1} \Rightarrow U_{1} \neq V_{1} \cup W_{1}$$

$$\mathbb{P}_{2}, W_{2} \subseteq U_{2} : U_{2} = V_{2} \cup W_{2} \Rightarrow U_{2} \neq V_{2} \cup W_{2}$$

$$U = \bigcup_{i \in \{1,2\}} U_{i} = U_{1} \cup U_{2} \neq (V_{1} \cup W_{1}) \cup (V_{2} \cup W_{2})$$

$$= (V_{1} \cup V_{2}) \cup (W_{1} \cup W_{2})$$

$$= \left(\bigcup_{i \in \{1,2\}} V_{i}\right) \cup \left(\bigcup_{i \in \{1,2\}} W_{i}\right) = V \cup W$$

Since the union of open sets is open, and V_i, V_i are open, we can conclude that V, W is open. Hence we have that there does not exists open subset V, W such that $U = V \cup W$ with $U = \bigcup_{i \in \{1,2\}} U_i, V = \bigcup_{i \in \{1,2\}} V_i, W = \bigcup_{i \in \{1,2\}} W_i$

Induction step: $I = \{1, \dots, n\}$

Assume that, $\bigcup_{i \in \{1,\dots,n\}} U_i$ is not connected. We want to show that $\bigcup_{j \in \{1,\dots,n+1\}} U_j$ is also connected.

Since U_{n+1} is a set in the family of connected subsets, we know that

$$\bigcup_{i \in \{1,\dots,n\}} \neq \left(\bigcup_{i \in \{1,\dots,n\}} V_i\right) \cup \left(\bigcup_{i \in \{1,\dots,n\}} W_i\right)$$

$$U_{n+1} \neq V_{n+1} \cup W_{n+1}$$

Where each V_i , V_{n+1} , W_i , W_{n+1} is open. Therefore, we have that

$$\bigcup_{j \in \{1, \dots, n+1\}} U_i = \left(\bigcup_{i \in \{1, \dots, n\}} U_i\right) \cup U_{n+1}$$

$$\neq \left(\bigcup_{i \in \{1, \dots, n\}} V_i\right) \cup \left(\bigcup_{i \in \{1, \dots, n\}} W_i\right) \cup \left(V_{n+1} \cup W_{n+1}\right)$$

$$= \left(\bigcup_{j \in \{1, \dots, n+1\}} V_j\right) \cup \left(\bigcup_{j \in \{1, \dots, n+1\}} W_j\right)$$

Since each V_j and W_j is open, it follows that $\bigcup_{j \in \{1,\dots,n+1\}} V_i$ and $\bigcup_{j \in \{1,\dots,n+1\}} W_i$ are open. Therefore, we can conclude that there does not exists open set $\bigcup_{j \in \{1,\dots,n+1\}} V_i$ and $\bigcup_{j \in \{1,\dots,n+1\}} V_i$ and $\bigcup_{j \in \{1,\dots,n+1\}} V_j$ that

$$\bigcup_{j \in \{1,\dots,n+1\}} U_i = \left(\bigcup_{j \in \{1,\dots,n+1\}} V_i\right) \cup \left(\bigcup_{j \in \{1,\dots,n+1\}} W_i\right)$$

Hence $\bigcup U_i$ is connected.

 $j \in \{1,...,n+1\}$ Since I didn't fix n in the induction step, we know that this holds for any n hence proven by induction that $U = \bigcup U_i$ is connected.

I is (not) eountable.) finite tre inter

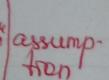
Problem 5

Let A and B be compact subsets of a Hausdorff space X. Prove that the intersection $A \cap B$ is compact in X.



Solution:

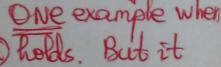
Assumption A, B compact on a Hausdorff space, then $A \cap B$ is not compact on a Hausdorff space.



I Will proof my assumption by a counterexample since then we have proven what is asked:

X is Hausdorff space, hence $\forall x, y \in X, \exists \text{ open } U, V \text{ s.t. } x \in U, y \in V \text{ and } U \cap V = \emptyset$. Choose A, B in such a way that $A \cap B = \{x_0 + r\}$ where x_0 is a boundary point. For example, let

 $A = \{a \in A : d(a, x_0) \le r\}$ $B = \{b \in B : d(b, x_0 + 2r) \le r\}$



We see that these sets are indeed compact.

When we take two points nonequal to x_0 , let's say x, y, then we can define U, V in the following way:

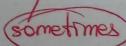
bots NOT prove that \Re we can define U, V in the ALWAYS hardius $\delta/2$. Define V by holds

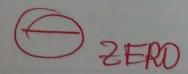
Let $\delta = d(x, y)$. Define U by the open ball, centered at x with radius $\delta/2$. Define V by the open ball centered at y with radius $\delta/2$.

We see that $x \in U, y \in V$ where U & V are open, with $U \cap V = \emptyset$, hence we found a counterexample for the Assumption.

Since I have found a counterexample for my assumption, I know that my assumption is incorrect, hence the given statement (in the question) must be correct.

sometimes





Problem 6

Let X be a consplete metric space and $f: X \to X$ be a mapping such that its rtime $T = f \circ ... \circ f$ is a Banach contraction (r > 1). Prove that f itself has a unique fixed point p in X. Inverse map does not exist. Is it about

Solution:

Use the notation: $f^{-k}(p) = f^{-1} \circ \dots \circ f^{-1}(p)$, which is a k time iteration.

 $f^{r}(p) = p$ $f^{r-1}(p) \Leftrightarrow f^{r-2}(p) = f^{-2}(p) \Leftrightarrow \dots \Leftrightarrow f^{1}(p) = f^{-(r-1)}(p) \Leftrightarrow p \Leftrightarrow f^{-r}(p)$ We know that $f^{-1}(x) = x$ for any value, since the inverse of the inverse is the original

again. Hence when r is odd we have that -(r-1) is even hence

$$f^{1}(p) = f^{-(r-1)}(p) = p$$

So we have indeed for r is odd that f itself has a unique fixed point. Unique since the banach contraction is unique, hance p is unique.

When r is even, we have that 1-r is odd, hence 2-r is even, which gives

$$f^{1}(p) = f^{-(r-1)}(p) = f^{-1}(f^{2-r}(p)) = f^{-1}(p)$$

So then we have that $f^1(p) = f^{-1}(p)$ which is only possible when f(p) = p. Therefore, when r is even, $f^r(p) = p \Rightarrow f(p) = p$ so a unique fixed point p in X.

Since I have proven that $f^r(p) = p \Rightarrow f(p) = p$ for r is even and r is odd gives a unique fixed point p in X, we can conclude that $f^r(p) = p$ implies that f itself has a unique fixed point p in X.

Consider R20 -> R20: collapse [0,1] to [0] and shaff x>1 to x-1.

Based on this example, it is easy to make fa contraction!