Resit Metric& topological spaces

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Problem 1:

Decide if the function

$$d(x,y) = \begin{cases} 0 \text{ if } x = y \\ x + y \text{ if } x \neq y \end{cases}$$

is or is not a metric on the set $\mathbb{N}_{\geq 1}, \mathbb{N}_{\geq 0}, \mathbb{N}_{\geq 0} \cup \{0\}$,on $\mathbb{R}_{\geq 0}$. If 'yes' at least once, then draw the respective open disks $B_{r=10}(x_0 = 2)$ and $B_{r=2}(x_0 = 10)$

Solution:

Def $\left\{ \begin{array}{l} \text{Note that } d(x,y) \text{ is a metric on the set } M \text{ if:} \\ 1)\ \forall x, y \in M : d(x,y) \geq 0 \text{ and } d(x,y) = 0 \Leftrightarrow x = y \\ 2)\ \forall x, y \in M : d(x,y) = d(y,x) \\ 3)\ \forall x, y, z \in M : d(x,z) \leq d(x,y) + d(y,z) \end{array} \right.$

Def $\{$ Open ball: $B_r(x_0) = \{x \in M : d(x, x_0) < r\}$

The set $\mathbb{N}_{\geq 1}$

Let $x = y$. We see that $d(x, y) = 0$.
Take $x \neq y$. So then we see that $d(x, y) = x + y$. Since $x \geq 1, y \geq 1$ we can conclude that $x + y \geq 2$ hence we have that $x \neq y \Rightarrow d(x, y) > 0$.
So for all values $x, y \in \mathbb{N}_{\geq 1}$ it follows that $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$ hence the first condition holds. ✓

Let $x = y$ so then we have that $d(x, y) = 0$ and $d(y, x) = 0$.
Take $x \neq y$. So then we have $d(x, y) = x + y$ and $d(y, x) = y + x$. Since $x + y = y + x$ it follows that $d(x, y) = d(y, x)$.
Hence it follows, that $\forall x, y \in \mathbb{N}_{\geq 1}$ we have $d(x, y) = d(y, x)$ so the second condition holds. ✓

Let $x = z$. So then we have that $d(x, z) = 0$. Since $d(x, y) \geq 0$ and $d(y, z) \geq 0$ (see first condition), it follows that $d(x, y) + d(y, z) \geq 0$. So,we have that $d(x, z) \leq d(x, y) + d(y, z)$. Note that this holds when $x = z = y$ and $x = z \neq y$. ✓
Let $x \neq z \neq y$. So then we have

$x = z = y$
$x = z \neq y$
$x = y$? ? $x \neq z \neq y$

$$d(x, z) \leq d(x, y) + d(y, z)$$
$$\Leftrightarrow x + z \leq (x + y) + (y + z) = x + 2y + z$$
$$\Leftrightarrow 0 \leq 2y$$

? or zero.

Since we have $y \in \mathbb{N}_{\geq 1}$ we know that $y > 0$ so $2y > 0$. Therefore it ~~follows~~ that $d(x, z) \leq d(x, y) + d(y, z)$ when $x \neq y \neq z$. not yet

So far, if $x \neq y$.

~~Hence~~ it follows, that $\forall x, y, z \in \mathbb{N}_{\geq 1}$ we have $d(x, z) \leq d(x, y) + d(y, z)$ so the third condition holds. Not yet.

Since all the three condition holds, ? not yet we ~~can~~ conclude that the function $d(x, y)$ is a metric on the set $\mathbb{N}_{\geq 1}$.

Since I use latex, I can not draw the open balls. But I am able to describe these open balls, by using the definition above:
$B_{r=10}(x_0 = 2) = \{x \in \mathbb{N}_{\geq 1} : d(x, 2) < 10\} = \{1, 2, 3, 4, 5, 6, 7\}$
$B_{r=2}(x_0 = 10) = \{x \in \mathbb{N}_{\geq 1} : d(x, 10) < 2\} = \{10\}$
This last open ball follows from the fact that $d(x_0, x_0) = 0 < r$ ✓

? $x \neq z = y$.

The set $\mathbb{N}_{\geq 0}$

Let $x = y$. We see that $d(x, y) = 0$.
Take $x \neq y$. So then we see that $d(x, y) = x + y$. Since $x \geq 0, y \geq 0$ but they are not simulatenous zero,we can conclude that $x + y \geq 1$ hence we have that $x \neq y \Rightarrow d(x, y) > 0$.
So for all values $x, y \in \mathbb{N}_{\geq 0}$ it follows that $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$ hence the first condition holds.

Let $x = y$ so then we have that $d(x, y) = 0$ and $d(y, x) = 0$.
Take $x \neq y$. So then we have $d(x, y) = x + y$ and $d(y, x) = y + x$. Since $x + y = y + x$ it follows that $d(x, y) = d(y, x)$.
Hence it follows, that $\forall x, y \in \mathbb{N}_{\geq 0}$ we have $d(x, y) = d(y, x)$ so the second condition holds.

Let $x = z$. So then we have that $d(x, z) = 0$. Since $d(x, y) \geq 0$ and $d(y, z) \geq 0$ (see first condition), it follows that $d(x, y) + d(y, z) \geq 0$. So,we have that $d(x, z) \leq d(x, y) + d(y, z)$. Note that this holds when $x = z = y$ and $x = z \neq y$.
Let $x \neq z \neq y$. So then we have

$x = z = y$
$x = z \neq y$
$x \neq z \neq y$!

$$d(x, z) \leq d(x, y) + d(y, z)$$
$$\Leftrightarrow x + z \leq (x + y) + (y + z) = x + 2y + z$$
$$\Leftrightarrow 0 \leq 2y$$

not always. ? $x = y$?

Since we have $y \in \mathbb{N}_{\geq 0}$ we know that $y \geq 0$ so $2y \geq 0$. ~~Therefore~~ it ~~follows~~ that $d(x, z) \leq d(x, y) + d(y, z)$ when $x \neq y \neq z$.
~~Hence~~ it follows, that $\forall x, y, z \in \mathbb{N}_{\geq 0}$ we ~~have~~ $d(x, z) \leq d(x, y) + d(y, z)$ so the third condition ~~holds~~. Not yet.

Since all the three condition holds, we can conclude that the function $d(x, y)$ is a metric on the set $\mathbb{N}_{\geq 0}$.

Since I use latex, I can not draw the open balls. But I am able to describe these open balls, by using the definition above:
$B_{r=10}(x_0 = 2) = \{x \in \mathbb{N}_{\geq 0} : d(x, 2) < 10\} = \{0, 1, 2, 3, 4, 5, 6, 7\}$ ✓
$B_{r=2}(x_0 = 10) = \{x \in \mathbb{N}_{\geq 0} : d(x, 10) < 2\} = \{10\}$
This last open ball follows from the fact that $d(x_0, x_0) = 0 < r$

? $x \neq z = y$.

The set $\mathbb{R}_{\geq 0}$

Let $x = y$. We see that $d(x, y) = 0$.
Take $x \neq y$. So then we see that $d(x, y) = x + y$. Since $x \geq 0, y \geq 0$ but they are not simulatenous zero,we can conclude that $x + y \geq 1$ hence we have that $x \neq y \Rightarrow d(x, y) > 0$.
So for all values $x, y \in \mathbb{R}_{\geq 0}$ it follows that $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$ hence the first condition holds.

Let $x = y$ so then we have that $d(x, y) = 0$ and $d(y, x) = 0$.
Take $x \neq y$. So then we have $d(x, y) = x + y$ and $d(y, x) = y + x$. Since $x + y = y + x$ it follows that $d(x, y) = d(y, x)$.
Hence it follows, that $\forall x, y \in \mathbb{R}_{\geq 0}$ we have $d(x, y) = d(y, x)$ so the second condition holds.

Let $x = z$. So then we have that $d(x, z) = 0$. Since $d(x, y) \geq 0$ and $d(y, z) \geq 0$ (see first condition), it follows that $d(x, y) + d(y, z) \geq 0$. So,we have that $d(x, z) \leq d(x, y) + d(y, z)$. Note that this holds when $x = z = y$ and $x = z \neq y$.
Let $x \neq z \neq y$. So then we have

$x = z = y$ ✓
$x = z \neq y$ ✓,
$x \neq z \neq y$ ✓,

$$
\begin{aligned}
& d(x, z) \leq d(x, y) + d(y, z) \\
\Leftrightarrow & x + z \leq (x + y) + (y + z) = x + 2y + z \\
\Leftrightarrow & 0 \leq 2y
\end{aligned}
$$

not always.

Since we have $y \in \mathbb{R}_{\geq 0}$ we know that $y \geq 0$ so $2y \geq 0$. ~~Therefore~~ it follows that $d(x, z) \leq d(x, y) + d(y, z)$ when $x \neq y \neq z$.
~~Hence~~ it follows, that $\forall x, y, z \in \mathbb{R}_{\geq 0}$ we ~~have~~ $d(x, z) \leq d(x, y) + d(y, z)$ so the third condition ~~holds~~. not yet.

Since all the three condition holds ? we ~~can~~ conclude that the function $d(x, y)$ is a metric on the set $\mathbb{R}_{\geq 0}$.

Since I use latex, I can not draw the open balls. But I am able to describe these open balls, by using the definition above:
$B_{r=10}(x_0 = 2) = \{x \in \mathbb{R}_{\geq 0} : d(x, 2) < 10\} = [0, 8)$
$B_{r=2}(x_0 = 10) = \{x \in \mathbb{R}_{\geq 0} : d(x, 10) < 2\} = \{10\}$ ✓
This last open ball follows from the fact that $d(x_0, x_0) = 0 < r$

? $x \neq z = y$ ✓

$\Sigma = 3 \times 2pt + 1 \cdot 3 + 2 \times 4pt = 17pt.$

The set $\mathbb{N}_{\geq 0} \cup \{-1\}$

Take $x = -1$ and $y = 1$ (otherway around is also possible). Since $x \neq y$ it follows from the function that $d(x, y) = x + y = -1 + 1 = 0$. So we have that not $\forall x, y \in \mathbb{N}_{\geq 0} \cup \{-1\}$ it holds that $d(x, y) = 0 \Leftrightarrow x = y$.
Therefore, the first condition does not hold. Since all the three conditions must hold, we can conclude that our function $d(x, y)$ is not a metric on the set $\mathbb{N}_{\geq 0} \cup \{-1\}$ ✓

Problem 2:

Question a:

Show by a counterexample that the image $f(V)$ of a closed set $V \subseteq X$ under a continuous map $f : X \to Y$ is not necessarily closed in Y.

ZERO 0pt

Solution:
Take $X = \mathbb{R}$ with $V = [0, 2\pi]$. Let

full circle

Not defined on

$$f(x) = \tan(x - \pi/2) = \frac{\sin(x - \pi/2)}{\cos(x - \pi/2)}$$

undef if $x - \frac{\pi}{2} = \frac{\pi}{2}$, $x = \pi \in [0, 2\pi]$.

We ~~see~~ that this function has ~~a limit~~, so is not closed, so Y is not closed.

Question b:

Show by a counterexample that the image $f(U)$ of an open set $U \subseteq X$ under a continuous map $f : X \to Y$ is not necessarily open in Y.

Solution:
Let X be the set of real numbers. Take U to be an open subset of X. Let $x \in U$. Define $f : X \to Y$ such that

which topology? def? what is Y?

$Y = ?$ $\mathcal{T}_X = ?$ $\mathcal{T}_Y = ?$

$$f(x) = \begin{cases} 0 \text{ if } x < 0 \\ 1 \text{ if } x \geq 0 \end{cases}$$

Our ~~set~~ of U is open while our set $f(U) \in \{0, 1\}$ which is obviously not open. We ~~see~~ that $f(x)$ is continuous, ~~since~~ for every $y \in f(U)$ there exists at least one $x \in U$ such that $f(x) = y$.

wrt which topology? so what?

recall def

0pt ZERO

The set I not necessarily countable.

Problem 4:

Let (X, d_x) be a metic space and $\{U_i | i \in I\}$ be a family of connected subsets $U_i \subseteq X$ such that $U_i \cap U_j \neq \emptyset$ for all $i, j \in I$. Prove that the union $U = \bigcup_{i \in I} U_i$ is connected.

?

Solution:
Since U_i is ~~not~~ connected, we ~~know~~ that there does not exists open sets $V_i, W_i \subseteq U_i$ such that $U_i = V_i \cup W_i$.
I will prove that this also holds for U, by induction ? $V_i := \emptyset$, $U_i := W_i$ if U_i open (!)
Base case: $I = \{1, 2\}$.
We know that $\{U_i | i \in I\}$ is a family of connected subsets. Therefore, we ~~have~~ that

$$\cancel{\nexists V_1, W_1} \subseteq U_1 : U_1 = V_1 \cup W_1 \Rightarrow U_1 \neq V_1 \cup W_1$$
$$\cancel{\nexists V_2, W_2} \subseteq U_2 : U_2 = V_2 \cup W_2 \Rightarrow U_2 \neq V_2 \cup W_2$$

$$\begin{aligned} U = \bigcup_{i \in \{1,2\}} U_i = U_1 \cup U_2 &\neq (V_1 \cup W_1) \cup (V_2 \cup W_2) \\ &= (V_1 \cup V_2) \cup (W_1 \cup W_2) \\ &= \left(\bigcup_{i \in \{1,2\}} V_i \right) \cup \left(\bigcup_{i \in \{1,2\}} W_i \right) = V \cup W \end{aligned}$$

Since the union of open sets is open, and V_i, W_i are open, we can conclude that V, W is open.Hence we have that there does not exists open subset V, W such that $U = V \cup W$ with $U = \bigcup_{i \in \{1,2\}} U_i, V = \bigcup_{i \in \{1,2\}} V_i, W = \bigcup_{i \in \{1,2\}} W_i$
Induction step: $I = \{1, \ldots, n\}$
Assume that, $\bigcup_{i \in \{1,\ldots,n\}} U_i$ is not connected. We want to show that $\bigcup_{j \in \{1,\ldots,n+1\}} U_j$ is also connected.
Since U_{n+1} is a set in the family of connected subsets, we know that

$$\bigcup_{i \in \{1,\ldots,n\}} \neq \left(\bigcup_{i \in \{1,\ldots,n\}} V_i \right) \cup \left(\bigcup_{i \in \{1,\ldots,n\}} W_i \right)$$
$$U_{n+1} \neq V_{n+1} \cup W_{n+1}$$

Where each $V_i, V_{n+1}, W_i, W_{n+1}$ is open. Therefore, we have that

$$\begin{aligned}
\bigcup_{j\in\{1,\ldots,n+1\}} U_i &= \left(\bigcup_{i\in\{1,\ldots,n\}} U_i\right) \cup U_{n+1} \\
&\neq \left(\bigcup_{i\in\{1,\ldots,n\}} V_i\right) \cup \left(\bigcup_{i\in\{1,\ldots,n\}} W_i\right) \cup (V_{n+1} \cup W_{n+1}) \\
&= \left(\bigcup_{j\in\{1,\ldots,n+1\}} V_j\right) \cup \left(\bigcup_{j\{1,\ldots,n+1\}} W_j\right)
\end{aligned}$$

Since each V_j and W_j is open, it follows that $\bigcup_{j\in\{1,\ldots,n+1\}} V_i$ and $\bigcup_{j\in\{1,\ldots,n+1\}} W_i$ are open.

Therefore, we can conclude that there does not exists open set $\bigcup_{j\in\{1,\ldots,n+1\}} V_i$ and $\bigcup_{j\in\{1,\ldots,n+1\}} W_i$ such that

$$\bigcup_{j\in\{1,\ldots,n+1\}} U_i = \left(\bigcup_{j\in\{1,\ldots,n+1\}} V_i\right) \cup \left(\bigcup_{j\in\{1,\ldots,n+1\}} W_i\right)$$

Hence $\bigcup_{j\in\{1,\ldots,n+1\}} U_j$ is connected.

Since I didn't fix n in the induction step, we know that this holds for any n hence proven by induction that $U = \bigcup_{i\in I} U_i$ is connected.

finite +ve integer

I is not countable.

0

ZERO

Problem 5

Let A and B be compact subsets of a Hausdorff space X. Prove that the intersection $A \cap B$ is compact in X.

(*)

Solution:

Assumption A, B compact on a Hausdorff space, then $A \cap B$ is not compact on a Hausdorff space.

one counterexample to a "false" assumption

I Will proof my assumption by a counterexample since then we have proven what is asked:

DOES give ONE example when (*) holds. But it DOES NOT prove that (*) ALWAYS holds.

X is Hausdorff space, hence $\forall x, y \in X, \exists$ open U, V s.t. $x \in U, y \in V$ and $U \cap V = \emptyset$. Choose A, B in such a way that $A \cap B = \{x_0 + r\}$ where x_0 is a boundary point. For example,let

what does this mean?

$$A = \{a \in A : d(a, x_0) \leq r\}$$
$$B = \{b \in B : d(b, x_0 + 2r) \leq r\}$$

We see that these sets are indeed compact.
When we take two points nonequal to x_0 ,let's say x, y, then we can define U, V in the following way:
Let $\delta = d(x, y)$. Define U by the open ball, centered at x with radius $\delta/2$. Define V by the open ball centered at y with radius $\delta/2$.
We see that $x \in U, y \in V$ where $U \& V$ are open, with $U \cap V = \emptyset$, hence we found a counterexample for the Assumption.

Since I have found a counterexample for my assumption, I know that my assumption is incorrect, ~~hence~~ the given statement (in the question) must be correct.

sometimes

sometimes

⊖ ZERO

Problem 6

iteration

Let X be a complete metric space and $f : X \to X$ be a mapping such that its r-time $f^r = f \circ \ldots \circ f$ is a Banach contraction ($r > 1$). Prove that f itself has a unique fixed point p in X.

Inverse map does not exist. Is it about preimages?!

Solution:

Use the notation: $f^{-k}(p) = f^{-1} \circ \ldots \circ f^{-1}(p)$,which is a k time iteration.

$$f^r(p) = p \Leftrightarrow f^{r-1}(p) = f^{-1}(p) \Leftrightarrow f^{r-2}(p) = f^{-2}(p) \Leftrightarrow \ldots \Leftrightarrow f^1(p) = f^{-(r-1)}(p) \Leftrightarrow p = f^{-r}(p)$$

one point many points.

We know that $f^{-2}(x) = x$ for any value, since the inverse of the inverse is the original again. Hence when r is odd we have that $-(r-1)$ is even hence

$$f^1(p) = f^{-(r-1)}(p) = p$$

So we have indeed for r is odd that f itself has a unique fixed point. Unique since the banach contraction is unique, hence p is unique.
When r is even, we have that $1 - r$ is odd, hence $2 - r$ is even, which gives

$$f^1(p) = f^{-(r-1)}(p) = f^{-1}(f^{2-r}(p)) = f^{-1}(p)$$

So then we have that $f^1(p) = f^{-1}(p)$ which is only possible when $f(p) = p$. Therefore, when r is even, $f^r(p) = p \Rightarrow f(p) = p$ so a unique fixed point p in X.

Since I have proven that $f^r(p) = p \Rightarrow f(p) = p$ for r is even and r is odd gives a unique fixed point p in X, we can conclude that $f^r(p) = p$ implies that f itself has a unique fixed point p in X.

ZERO.

Consider $\mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$: collapse $[0,1]$ to $\{0\}$ and shift $x \geq 1$ to $x-1$.

$p=0$ is a fixed point, but f is NOT invertible.

Based on this example, it is easy to make f a contraction!